Part 2

Introduction to Mathematical Methods in Physics(*)

Mathieu Aurousseau (University of Johannesburg)

mathieu.aurousseau@cern.ch

(*) Not all of them

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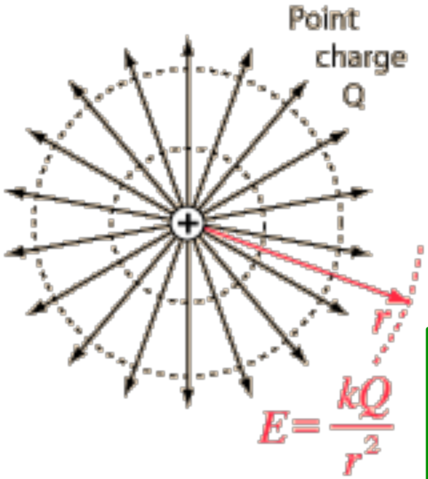
DIRAC DISTRIBUTION AND GREEN'S FUNCTION

At the end of this lectures, you should …

- understand the concept of generalized function
- know how to use a Dirac distribution
- understand what is a Green's function of an operator
- realize how important Klein-Gordon is …

A physics problem …



$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}\,\vec{e}_r$$

$$\nabla \cdot \vec{E} = 0$$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0}$$

$$\int_V (\nabla \cdot \vec{v}) d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

(Gauss theorem)

Contradiction ??

A point-like source has a charge located on a "0 volume" : infinite charge density on the point, and 0 elsewhere …

=> we want something that it infinite in 1 point and 0 elsewhere : is there such a "function" ?

The Dirac delta "function"

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$$\delta(x) = \begin{cases} +\infty & \text{if } x = 0 \,, \\ 0 & \text{if } x \neq 0 \,. \end{cases} \tag{79}$$

and

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1 \quad . \tag{80}$$

=> can describe the density of charge of a point-like source !

$$f(x)\delta(x) = f(0)\delta(x) \qquad \int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$$

$$\delta(x-a) = \begin{cases} +\infty & \text{if } x = a \,, \\ 0 & \text{if } x \neq a \,, \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x-a)dx = 1$$

Is it really *a function ?*

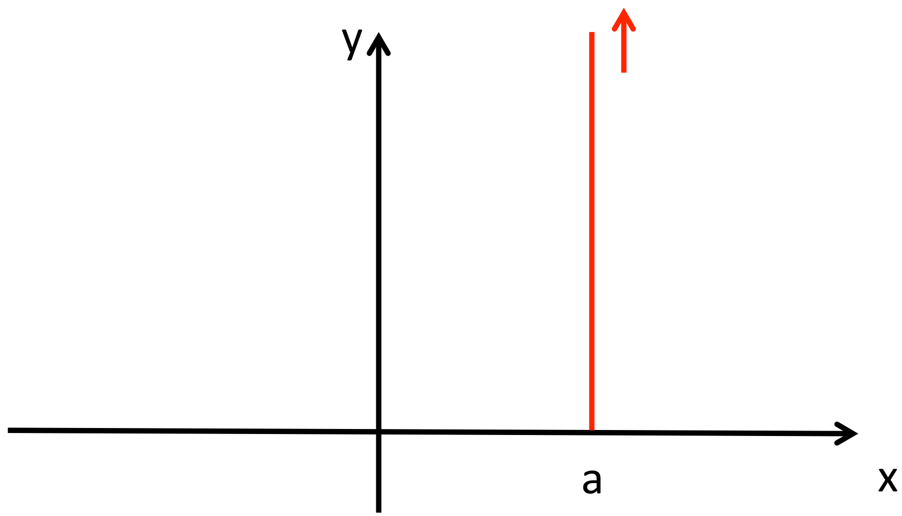
Whatever it is :

- it is not continuous
- it has an "infinite" point (in its own definition …)
- it seems to make sense only within an integral

In fact, it should be called : a "generalized function", or a "distribution"

Acts on a function, returns a number

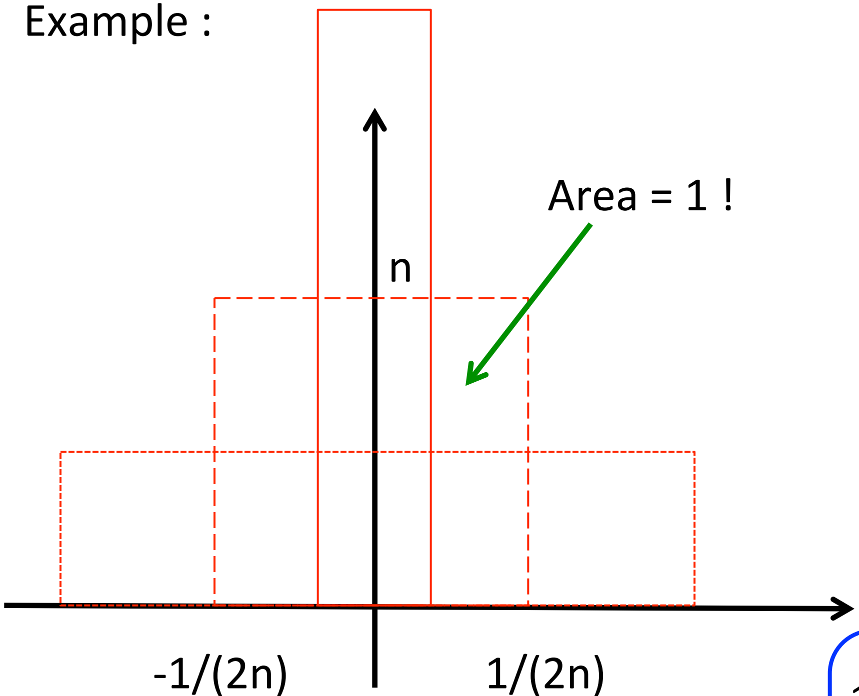
$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0)$$



How to construct it ?

The Dirac distribution can be considered as the limit of a sequence of functions

Example :



Other examples :

- Triangles
- Gaussians
- sin(x)/x
- …

Do it yourself !

1. Convince yourself that these sequences "converge" to the Dirac distribution
2. Check/prove that the Dirac distribution can be considered as the derivative of the Heaviside step function

Some examples

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Do it yourself !

1. Calculate the following integrals

$$\int_{-\infty}^{+\infty} \cos x \; \delta(x-\pi) dx$$

$$\int_{-1}^{+1} (x^2+1)\delta(x) dx$$

$$\int_{10}^{20} (x^2+1)\delta(x) dx$$

2. In a physics context : how would you write the energy and momentum conservation with a Dirac distribution ?

Identification of a Delta function

The particular nature of the Dirac Delta function makes it slightly more difficult to identify. In particular, two expressions involving "Delta" functions, D_1 and D_2, they are considered equal if:

$$\forall f, \int_{-\infty}^{+\infty} f(x) D_1(x) dx = \int_{-\infty}^{+\infty} f(x) D_2(x) dx \quad . \tag{86}$$

It is important that the above equation be valid for *any* "well-behaved" function f.

This definition can be used to prove for example that :

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Do it yourself !

1. Prove it !

Back to our problem …

$$\nabla \cdot \vec{E} = 0 \qquad\qquad \nabla \cdot \vec{E} = \frac{Q}{\varepsilon_0}\delta(\vec{r})$$

=> With this and the definition of the Dirac delta function, Gauss theorem holds !

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$$\mathcal{L}u(x) = f(x) \qquad \text{Eq. (1)}$$

Some linear differential operator

source

point-like source

$$\mathcal{L}G(x,x') = \delta(x - x')$$

Green's function associated to the linear differential operator

x = variable, x' = parameter

<u>A solution to the differential equation (1) is :</u>

$$u(x) = \int dx' G(x,x') f(x')$$

Do it yourself !

1. Check it !

What is the physical meaning of G ?

$$\mathcal{L}G(x,x') = \delta(x - x')$$

A source *f(x)* in general is a collection or continuum of sources
G is actually the solution of the equation for a point-like source ! Like solving a problem for every possible point …

Back to electrostatics …

$$\nabla \cdot \vec{E} = \frac{Q}{\varepsilon_0}\delta(\vec{r}) \qquad \vec{E} = -\nabla V$$

$$-\nabla^2\left(\frac{\varepsilon_0}{Q}V\right) = \delta(\vec{r})$$

The electric potential is the Green function G of the operator $-\nabla^2$

An example : our good old friend Klein-Gordon

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$$(\Box^2 + m^2)\psi = s(x)$$

$$\Box^2 \equiv \partial_t^2 - \nabla^2$$

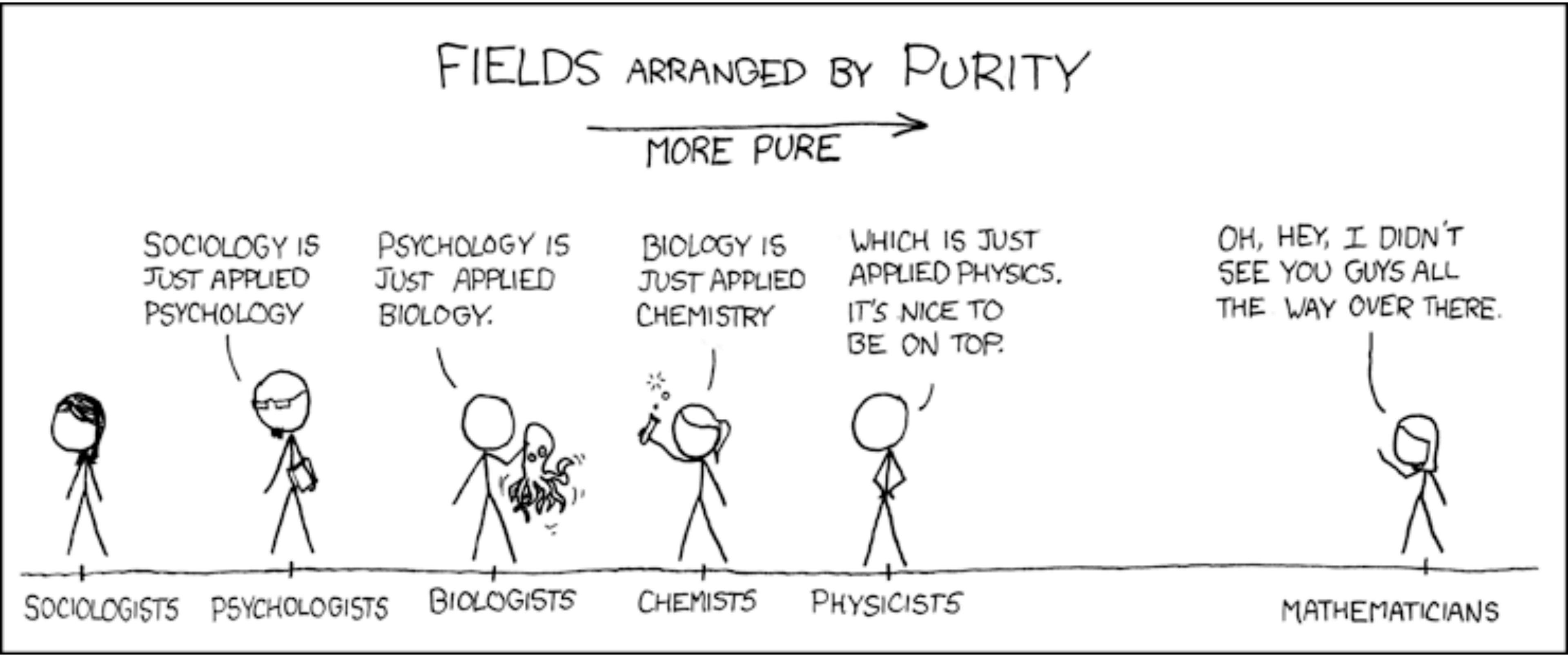
source

Linear differential operator

$$(\Box^2 + m^2)G(x, x') = \delta(x - x')$$

$$\tilde{G}(p) = -\frac{1}{p^2 - m^2}$$

Conclusions



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